DISSEMINATION
MotLang

Motives and the Langlands programme

Professor Timo Richarz talks to the PRj about the start of the MotLang project.

A decades-long lack of focus on tackling motives within the Langlands programme is about to be rectified by the MotLang project, thanks to a €1.5 million ERC Starting Grant awarded to Professor Timo Richarz.

The MotLang (Motives and the Langlands Program) project, led by Professor Timo Richarz at TU Darmstadt Department of Mathematics, is devoted to notoriously complex and multi-faceted areas of mathematics—the Langlands programme and the theory of motives. Richarz believes that now is the right time to combine these two areas.

“There are two different theories, the theory of motives on the one hand and the Langlands programme on the other hand,” explains Richarz. “They should be closely combined. The Langlands programme should talk about motives—this was postulated by Langlands from the beginning, but somehow the focus shifted. Now is the time to combine these two areas.”

What is the Langlands programme?

The Langlands programme is one of the biggest projects in modern mathematical research. Its aim? To find laws that govern the behaviour of solution sets of polynomial equations. The programme has the potential to bring us one step closer to verifying the Langlands conjecture about the relationship between motives and automorphic forms, helping to solve long-standing, complex mathematical problems.

The programme is a furtherance of the work of Canadian mathematician Robert Langlands. In the 1960s, Langlands first suggested deep connections between harmonic analysis and number theory. Since then, hundreds of the highest-calibre mathematicians of our time have devoted their work to proving Langlands’ conjectures.

Harmonic analysis and number theory are completely different fields of mathematics. They have different concepts, languages and techniques. There was seemingly no relationship between the two until Langlands conjecture of a close connection between motives, a linearisation of solution sets of polynomial equations and automorphic forms.

Harmonic analysis and automorphic forms

Hamonic analysis is all about curves, waves and symmetries. Think of sound, where sine waves are the most simple waveform and are the building blocks of more complex sounds. Sine waves can be added together to create periodic waves that have peaks and troughs and repeat over and over again. If a violin plays the note of A, the sound will be made up of the harmonics of A, the various sine waves that make up that sound. Hence the study of periodic waves is called ‘harmonic analysis’.

Automorphic forms are a concept within harmonic analysis that express the general concept of periodic waves using advanced geometrical language.

Number theory, polynomial equations and Galois

Number theory deals with, unsurprisingly, numbers: whole numbers, rational numbers, real numbers etc. It’s a world of algebra and equations. Finding solutions to **polynomial equations** is the most basic concern within the Langlands programme.

In order to solve a polynomial equation in a variable x, you must calculate its roots; the value(s) of x that makes the polynomial equal 0. Lower-power polynomials, those with the highest power up to 2, are easy and are known to a lot of people from their school days; for 3 and 4, solutions have existed since the sixteenth century, but it’s still difficult. The higher-power polynomials, those with variables to the power of 5 or more, proved significantly more problematic for mathematicians.

Indeed, in 1832 a French mathematician by the name of Évariste **Galois** proved that formulas to solve the higher-power polynomials do not exist. Galois devised a different approach. Instead of calculating the roots of polynomials, we should study the relationships between them.

> “Go to the roots of these calculations! Group the operations. Classify them according to their complexities rather than their appearances! This, I believe, is the mission of future mathematicians. This is the road on which I am embarking in this work.”
>
> Évariste Galois

Galois furthered his proposal for studying the symmetries between roots by developing a mathematical object called **Galois groups**. These Galois groups describe the complicated algebraic relationships between the roots of polynomials.

Motives

The theory of **motives** exists within algebraic geometry in an effort to unify a set of theories, so-called cohomology theories, that typically relate to topology. The idea of motives was immature at the time of Langlands conjectures, and therefore mathematicians resorted to studying a proxy of motives, namely representations of Galois groups. Although significant advances in both the theory of motives and the Langlands programme have been made since, there has been little progress in identifying a direct link between motives and automorphic forms.

A shift in focus

Richarz is keen to highlight a historical shift in focus away from motives within the Langlands programme. "Mathematicians had problems to implement and found other solutions to establishing the Langlands programme. The focus shifted away from the theory of motives, at least from the point of view of the programme.

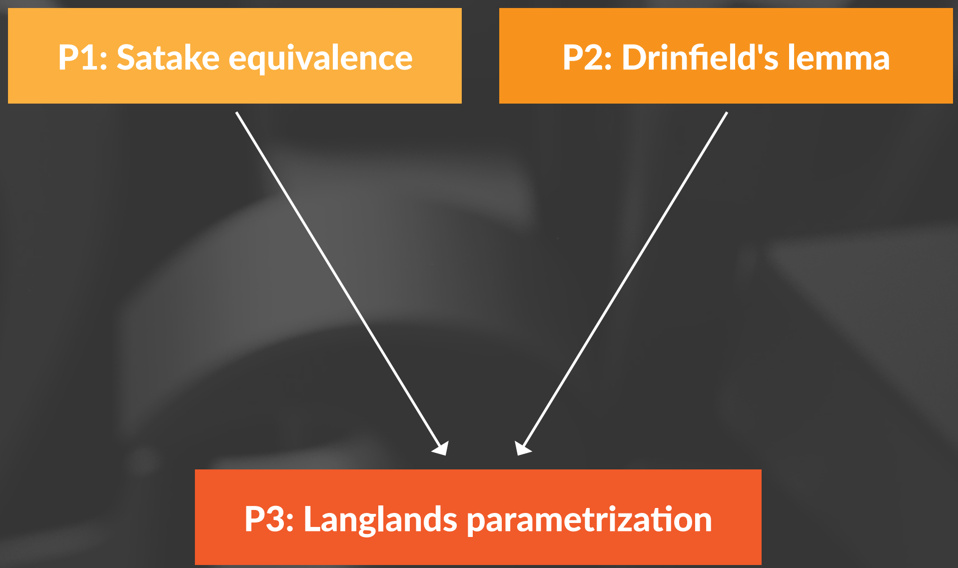
I believe that we should explore applications of the modern theory of motives within the Langlands programme.

"To really calculate the Langlands correspondents and write down explicit calculations, there are other tools which seem more appropriate. The focus shifted somehow. People on the Langlands programme were happy using other techniques that, from a theoretical perspective, have drawbacks.

"We are now at a point where we can combine these theories. With breakthroughs in both theories, developed independently, I thought it was about time someone got started on this."

The MotLang project

MotLang seeks to make a breakthrough in understanding the relationship between automorphic forms and motives for (algebraic) function fields. The project, which started in April of 2022, consists of three individual smaller projects.



Project 1: Satake equivalence

The geometric Satake equivalence is a version of the Satake isomorphism—a theorem that deals with p-adic groups and plays a pivotal role in the arithmetic Langlands conjectures.

The Satake equivalence is an important tool in geometric representation theory and has a wide range of applications. Consequently, it is well known within the mathematics community, with many mathematicians working in that direction.

"The aim here is to do such a thing for motives," says Richarz, "to improve on the motivic Satake equivalence from my joint work with Jakob Scholbach, which will feed into the final project: Langlands parametrisation."

Project 2: Drinfeld's lemma

The Drinfeld's lemma project is a particular passion for Richarz. Why? "It's hard," says Richarz, who clearly relishes the challenge. "It somehow describes how the right cycles should break up on a product of varieties. Think about a plane; in that plane, you have a geometric figure described by polynomial equations. This lemma is supposed to give a criterion when this geometric figure is somehow parallel to one of the axes.

"The picture I had in mind seemed simple, but it is not so simple to implement. This stage of the project is about formulating a similar result in the language of motives. This is one of the things that I am working on right now."

Project 3: Langlands parametrisation

The third and final project within MotLang is Langlands parametrisation. It will combine the outcomes of the previous two projects to implement this motivic Langlands parametrisation.

"Mathematically, when the project is complete, we will get a step closer to verifying the Langlands conjecture about the relationship between motives of automorphic forms for function fields," says Richarz.

Conclusion

When asked to describe the impact of MotLang to 'the real world', Richarz is initially a little hesitant.

"Real world outcomes? Well, it's fundamental research. I have no applications intended as stated in the project proposal." And then Richarz's tone shifts to one of excitement. "But think about the mathematics developed at the beginning of the twentieth century. It is now running in the background of our Zoom meeting; it's working to keep our meeting secure. No one at that time envisaged Zoom or Amazon. Who knows what the future could hold!

"Mathematically, when the project is complete, we will get a step closer to verifying the Langlands conjecture about the relationship between motives and automorphic forms for function fields."

MotLang is set to return to the PRj at the conclusion of the project. Until then, you can follow the progress of Richarz and the MotLang project via https://timo-richarz.com

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PROJECT NAME

Motives and the Langlands Program (MotLang)

PROJECT PARTNERS

The MotLang project is based at the Mathematics Department of the Technical University of Darmstadt (TUDa), collaborating with a group from Padova (Unipd).

PROJECT LEAD PROFILE

Timo Richarz received his PhD from Bonn University and later worked at the University of Duisburg-Essen and Sorbonne Université in Paris. Since 2019 he has been a professor at TU Darmstadt for number theory and algebraic geometry. As principal investigator, he is involved in the collaborative research centre CRC326 GAUS and the ERC Starting Grant MotLang.

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