Convergence for nonconventional ergodic theorems

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The programme is funded by Professor Krause's New Investigator Grant on Pointwise Ergodic Theory, and he is also my supervisor during my stay in Bristol. My interests span the *pointwise ergodic theory*, *analytic number theory* and *additive combinatorics*. Previously, I studied at the University of Bonn (West Germany) and Jagiellonian University in Kraków (Poland).

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Background

The first theorem that describes convergence behaviour of pointwise averages was obtained by George David Birkhoff (1931). Its statement can be presented as:

Theorem 1. *Suppose that (X, μ, T) is a measure-preserving system and $f \in L^1(X)$. Then for almost all points x in X^1, one has that:*

$$\lim_{n\to\infty} \frac{1}{N} \sum_{n\leq N} f(T^n x)$$

exists. On top of that, when transformation is ergodic, then for almost all points $x \in X$ limit is equal to:

$$\int f \, d\mu$$

Birkhoff was motivated to establish this theorem from studying dynamical systems arising from differential equations on a smooth manifold. Later on, mathematicians discovered many applications of that statement. For instance, Furstenberg (1977), Katznelson and Ornstein used that to prove Szemerédi theorem in ergodic theory language:

Theorem 2. *Let (X, μ, T) be a measure-preserving system. Then, for every nonnegative $ff \in L^\infty(X)$ with positive spatial average, the limit:*

$$\liminf_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} \int_X fT^n f \ldots T^{(k-1)n} f \, d\mu$$

is greater than 0.

This theorem provides a major impetus to the serious development of additive combinatorics in the 20th century (see, for instance, Host and Kra [2005] and Green and Tao [2008]).

Later on, Bourgain (once he was advised by Furstenberg and Bellow) started research on pointwise ergodic theorems for nonconventional averages. He accomplished firstly that:

Theorem 3. *Let (X, μ, T) be a measure-preserving system, P an arbitrary polynomial with integer coefficients and $f \in L^p(X)$* some $p > 1$. *Then the averages:*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} T^{P(n)} f(x)$$

converge for almost every $x \in X$.

In the penultimate section of his 1989 work, Bourgain suggested obtaining the same result for a sequence of times $(\lfloor P(n) \rfloor)_{n=0}^{\infty}$, where $P \in \mathbb{R}[x]$. One of the my objectives during graduate studies will be to establish that convergence.

Another example of a nonconventional ergodic theorem with a single test function was obtained by Wierdl in 1988:

Theorem 4. *Let (X, μ, T) be a measure-preserving system and $f \in L^p(X)$ for some $p > 1$. Then the averages:*

$$\lim_{N\to\infty} \frac{1}{|\mathbb{P} \cap [N]|} \sum_{p\in\mathbb{P}\cap[N]} T^p f(x)$$

converge for almost every $x \in X$.

Number theorists investigated many arithmetic configurations on prime numbers. For instance, Euler, in the middle of 18th century, showed that every prime $p \equiv 1 \pmod 4$ can be written as $x^2 + y^2$. Later on, Weber (1882) made heavy use of tools from algebraic number theory to prove that for any positive integer u there are infinitely many primes of the form $x^2 + uy^2$. Denoting the set of these primes as $\mathbb{P}_u$, then $\mathbb{P}_u$ also has *good averaging properties* in the sense that:

Theorem 5. *Let (X, μ, T) be a measure-preserving system and $f \in L^p(X)$ for some $p > 1$. Then the averages:*

$$\lim_{N\to\infty} \frac{1}{|\mathbb{P}_u \cap [N]|} \sum_{p\in\mathbb{P}_u\cap[N]} T^p f(x)$$

converge for almost every $x \in X$.

This is the first result that I've obtained under the support of a Professor Krause Grant.

Pointwise convergence of bilinear ergodic averages

The aforementioned Szemerédi theorem 2 invited ergodic theory specialists to investigate different versions of multiple ergodic averages. Contrary to the grant topic proposal, the norm convergence for many of these averages is understood well; perhaps Furstenberg and Weiss (1996) and Walsh (2012) may serve as good references in that area. Meanwhile, pointwise convergence, in most cases, remains mysterious. The first significant result on this topic was realised by Jean Bourgain (1989):

Theorem 6. *Suppose that (X, μ, T) is a dynamical system and T_1, T_2 are powers of T. Then for $f1, f2 \in L^\infty(\mu)$:*

$$\frac{1}{N} \sum_{n=1}^{N} T_1^n f_1 T_2^n f_2$$

converges almost surely.

That was the only result for more than 30 years, until Krause, Mirek and Tao (2022) managed to obtain the seemingly simplest polynomial bilinear result.:

Theorem 7. *Suppose that (X, μ, T) is a dynamical system, $P \in \mathbb{Z}[x]$ is polynomial of degree at least 2 and $p_1, p_2, \in (1,\infty]$ are exponents satisfying $\frac{1}{p_1}+\frac{1}{p_2} \leq 1$. Sample two functions f_1, f_2 within respectively $L^{p_1}(X)$ and $L^{p_2}(X)$. Then:*

$$\frac{1}{N}\sum_{n=1}^{N} T_1^n f_1 T^{P(n)} f_2$$

converges almost surely.

The proof of that result relies on the work of Peluse and Prendiville (Peluse and Prendiville, 2022) within the additive combinatorics. In the future, together with two collaborators supported by the same grant (i.e. Dr Mousavi and Dr Sun) we intend to prove another result of that flavour:

Theorem 8. *Suppose that (X, μ, T) is a dynamical system, $\alpha, \beta \in \mathbb{Z}$ and $p_1, p_2 \in (1, \infty]$ are exponents satisfying $\frac{1}{p_1}+\frac{1}{p_2} \leq 1$. Sample two functions f_1, f_2 within respectively $L^{p_1}(X)$ and $L^{p_2}(X)$. Then:*

$$\frac{1}{|\mathbb{P} \cap [N]|}\sum_{p \in \mathbb{P} \cap [N]} T^{\alpha p} f_1 T^{\beta p} f_2$$

converges almost surely.

As far as I know, this theorem remains resistant to modern tools like double recurrence method exploited in (Krause, Mirek and Tao, 2022) or (Krause *et al.*, 2024). We believe, however that modern methods from analytic number theory (different approximations of von Mangoldt function in the Gowers norm or additive combinatorics (various inverse theorems like those in [Peluse, 2020; Leng, Sah and Sawhney, 2024]) together with simplified approach to 6 will be sufficient for the actual proof of 8.

Additive combinatorics

I am lucky enough to start my cooperation with Dr Sean Prendiville from Lancaster University. He is an author of the result regarding quantitative bound for sets avoiding polynomial patterns in $[N]$ (see Peluse and Prendiville[2022]).

Theorem 9. *There exists $c > 0$ such that any subset of $\{1, \dots, N\}$ of density at least $(\log \log N)^{-c}$ contains a nontrivial progression of the form $x, x + y, x + y^2$.*

My subjective is to obtain different quantitative bounds of that flavour in two different settings (i.e. either for subsets of $\mathbb{Z}$ or subsets of finite characteristic Dedekind domains $\mathbb{F}q[T]$). One of most technical propositions in (Peluse and Prendiville, 2022) shows that trilinear correlations evaluated on respectively $x, x + y, x + y^2$ can be locally controlled by Gowers norms. Recent progress in that topic shows that more sophisticated progressions (like those described in Peluse, Sah and Sawhney [2023], Kuca [2019] or Matthiesen, Teräväinen and Wang [2024]) can also be controlled by different norms that originate from Gowers norm. For instance, in the paper (Kravitz, Kuca and Leng, 2019), authors established that patterns of form $(x, y), (x + P(z), y), (x, y + P(z))$ (where $P \in \mathbb{Z}[x]$ is polynomial with an integer root of multiplicity 1) can be controlled by *Peluse-Gowers box norms* with small boxes. It seems that current understanding of these norms is rather unsatisfactory and scholars from different areas look forward to further rise of this theory. I wish I could be one of the people that manage to accomplish a bit in that regard.

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PCMEA
POINTWISE CONVERGENCE OF MULTIPLE ERGODIC AVERAGES

PROJECT SUMMARY

Given an arbitrary measure preserving system we show that the multilinear ergodic averages sampled along an arbitrary number of sequences coming from a Hardy field converge pointwise almost everywhere. We aim to prove this for as wide a class of Hardy field functions as possible. To do so, we establish a long variational inequality along lacunary sequences which implies a maximal inequality, norm convergence, and pointwise convergence.

By a transference argument it suffices prove this long variational inequality in the case that the measure preserving system is the integers. This reduction allows us to use tools from discrete harmonic analysis, additive combinatorics, and analytic number theory. We then give applications in areas such as upcrossings, equidistribution, and combinatorics.

PROJECT LEAD

Max obtained his master's in mathematics (MMath) at the University of Warwick in 2021 and is now a third year PhD student studying under the supervision of Professor Ben Krause (Bristol University). His research interests lie in harmonic analysis (discrete harmonic analysis in particular), pointwise ergodic theory, and additive combinatorics.

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