Pointwise convergence of multiple ergodic averages – PCMEA

Max O'Keeffe

The study of multiple ergodic averages has a long and rich history, dating back at least to H. Furstenberg's ergodic theoretic proof of Szemerédi's theorem.

Adobe Stock © Dr_Microbe

Pointwise convergence of multiple ergodic averages along Hardy field orbits

Consider a dynamical system (X, T) so that X is a set and $T : X \to X$ is some map. Since T maps X to itself, we can consider its iterates T^n for positive integers n, where T^n denotes the composition of T with itself n times. One is often interested in studying the orbits

$$\mathcal{O}(x) = \{T^n(x) : n \in \mathbb{N}\}$$

of elements $x \in X$. Typical questions are whether $\mathrm{o}(x)$ is a finite or infinite set or whether it is dense in X (if X is a topological space).

Understanding the orbit of every element $x \in X$ might be possible in some cases but is typically too much to ask. One solution is to endow X with a measure μ and to try to ask probabilistic questions about orbits or to try to understand the orbit of almost every $x \in X$. Ergodic theory seeks to answer these questions. A typical motivating question is to ask whether an orbit is equidistributed (with respect to μ) in X. For a subset A of X, the proportion of the first N points of the sequence $(T^n(x))_{n \in N}$ that lie in A is given by

$$\frac{|\{1 \leq n \leq N : T^n(x) \in A\}|}{N}.$$

Equidistribution asks how this quantity is related to $\mu(A)$, the probability of randomly drawing an element of A, as N grows large. More precisely, the sequence $(T^n(x))_{n \in N}$ is equidistributed if

$$\lim_{N\to\infty} \frac{|\{1 \leq n \leq N : T^n(x) \in A\}|}{N} = \mu(A)$$

holds for all (measurable) subsets A. Ergodic theory provides a condition for when the orbits of almost every x are equidistributed. Indeed, first note that the equidistribution condition can be rewritten as

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} 1_A(T^n(x)) = \int_X 1_A \, d\mu$$

where 1_A is the characteristic function of A, so that $1_A(x)$ equals 1 if $x \in A$ and equals 0 otherwise. One of the central results of ergodic theory is Birkhoff's pointwise ergodic theorem (Birkhoff, 1931). To state it, we will need some definitions. We say that T is measure preserving if $\mu(T^{-1}(A)) = \mu(A)$ for all measurable A, and a measure preserving map is called ergodic if any A satisfying $T^{-1}A = A$ up to sets of μ- measure 0 equal to zero or one. Informally, T is ergodic if it is "sufficiently randomising" (Krause, 2022), since the only sets which are invariant under T are equal, up to sets of measure zero, to the whole set or the empty set.

Theorem (Birkhoff). If $T : X \to X$ *is measure preserving, then for any* $f \in L^1(X)$ *the ergodic averages*

$$\frac{1}{N} \sum_{n=1}^{N} f(T^n(x))$$

converge for almost every $x \in X$. *Moreover, if T is ergodic, then the limit is* $\int_X f \, d\mu$.

One often interprets Bikhoff's theorem as saying that the time averages of any absolutely integrable f converge almost everywhere to the spatial average. Specialising $f = 1_A$, we see that if T is ergodic then the orbit of almost every $x \in X$ is equidistributed.

Birkhoff's theorem is open to generalisation. Indeed, one might be interested in the proportion of the first N points of the sequence $(T^{p(n)}(x))$ that lie in a subset of X, where $P : \mathrm{N} \to \mathrm{N}$ is some other sequence. Even more generally, we could fix subsets $A_1, \ldots, A_m$ and sequences $P_1, \ldots, P_m$ and ask when we have $T^{p(n)}(x) \in A_i$ simultaneously for all $1 \leq i \leq m$. Answering these questions would involve studying "non-conventional ergodic averages" (Furstenberg, 1990) of the form

$$A_N^{P_1,\ldots,P_m}(f_1, \ldots, f_m)(x) = \frac{1}{N} \sum_{n=1}^{N} \prod_{i=1}^{m} f_i(T^{P_i(n)}(x))$$

for measurable functions $f_1, \ldots, f_m : X \to \mathrm{C}$. We will see shortly that establishing pointwise convergence of such averages has been historically difficult. However, if one views these averages as a sequence of functions, then other notions of convergence become available. Indeed, we have the following counterpart to Birkhoff's theorem due to von Neumann, known as the mean ergodic theorem (von Neumann, 1932).

Theorem. *If* $T : X \to X$ *is measure preserving and* $f \in L^\infty(X)$, *then the sequence*

$$\frac{1}{N} \sum_{n=1}^{N} f \circ T^n$$

of elements of the Hilbert space $L^2(X)$ *converge with respect to the* $L^2(X)$ *norm. Moreover, if* **T** *is ergodic, then the limiting function is* $\int_X f \, d\mu$.

Thanks to a correspondence principle of Furstenberg (1977), one can deduce the existence of patterns of the form $\{x, x + P_1(n), \ldots, P_m(n)\}$ in 'large' subsets of the integers by proving

that $A_N^{P_1,\dots,P_m}(f_1,\dots,f_m)$ converges in $L^2(X)$ norm for any bounded functions $f_1,\dots,f_m$. The main area of focus was to take $P_1,\dots,P_m$ to be polynomials. Many results in this direction were proved (Furstenberg, 1977; Bergelson, 1987; Furstenberg and Weiss, 1996; Host and Kra, 2005), and this particular area of investigation culminated in 2005 when Leibman proved that the ergodic averages $A_N^{P_1,\dots,P_m}(f_1,\dots,f_m)$ converge in $L^2(X)$ norm for any polynomials $P_1,\dots,P_m$ (Leibman, 2005).

On the other hand, results about the pointwise convergence of $A_N^{P_1,\dots,P_m}(f_1,\dots,f_m)$ are much more limited. Bourgain had the first breakthrough in 1988 and 1989 in a series of papers (Bourgain, 1988a; 1988b; 1989) in which he showed that $A_N^P(f)$ converges almost everywhere whenever P is a polynomial of degree at least 2 and where $f \in L^p(X)$ with $1 < p < \infty$. In 1990 he then proved that for any $f, g \in L^\infty(X)$ the averages

$$\frac{1}{N}\sum_{n=1}^{N} f(T^n(x))g(T^{kn}(x))$$

converge for any integer k not equal to 0 or 1 (Bourgain, 1990). In terms of pointwise convergence results pertaining to $A_N^{P_1,\dots,P_m}(f_1,\dots,f_m)$ where $P_1,\dots,P_m$ are polynomials, nothing else was known until 2022 when Krause, Mirek, and Tao proved (Krause, Mirek and Tao, 2022) that the averages

$$\frac{1}{N}\sum_{n=1}^{N} f(T^n(x))g(T^{P(n)}(x))$$

converge almost everywhere for any polynomial P of degree at least 2 and any $f \in L^{p_1}(X)$ and $g \in L^{p_2}(X)$ with $1 < p_1, p_2 < \infty$ and $\frac{1}{p_1} + \frac{1}{p_2} \leq 1$.

Another direction one can take when studying ergodic averages is to consider sequences of the form $\lfloor P_1(n) \rfloor, \dots, \lfloor P_m(n) \rfloor$ where $P_1,\dots,P_m$ come from a *Hardy field*, which is informally the set of smooth functions such that they, and all their derivatives, are eventually monotonic. Here $\lfloor x \rfloor$ denotes the greatest integer less than x. As a concrete and illustrative example, take $P_i(t) = t^{c_i}$ where c_i is positive and non-integer. Wierdl began this line of investigation and proved (1989) that the averages

$$\frac{1}{N}\sum_{n=1}^{N} f(T^{\lfloor n^c \rfloor}(x))$$

converge almost everywhere for all $f \in L^p(X)$ with $p > 1$ whenever c is positive and non-integer. It was only in 2020 that Donoso, Koutsogiannis, and Sun (2020) were able to prove an extension, when they proved that the averages

$$\frac{1}{N}\sum_{n=1}^{N}\prod_{i=1}^{m} f_i(T^{\lfloor n^{c_i} \rfloor}(x))$$

converge pointwise for any $f_1,\dots,f_m \in L^\infty(X)$ and any $0 < c_1 < \dots < c_m \leq 1$. Since the exponents c_i are restricted to being less than 1, we see that the sequences $\lfloor n^{c_i} \rfloor$ are not sparse in this case. Thus Wierld's 1989 result remains the most general for sparse sequences.

The goal of my research is to prove that the averages

$$\frac{1}{N}\sum_{n=1}^{N}\prod_{i=1}^{m} f_i(T^{\lfloor n^{c_i} \rfloor}(x))$$

converge pointwise for any $f_i \in L^{p_i}(X)$ with $\frac{1}{p_1} + \dots + \frac{1}{p_m} \leq 1$ and any non-integer $0 < c1 < \dots < cm$.

Since the only pointwise convergence results for polynomials have at most two functions, this will represent the first pointwise convergence result for sparse sequences with more than two functions.

The insight, originating from probability theory and used by Bourgain (1988a), is to control a variation norm. Given $r > 0$, the r-variation norm of a sequence (a_N) is defined by

$$V^r(a_N) = \sup_{N\in\mathbb{N}} |a_N| + \sup_{J\in\mathbb{N}} \sup_{N_0<N_1<\dots<N_J} \left(\sum_{j=1}^{J} |a_{N_j} - a_{N_{j-1}}|^r\right)^{\frac{1}{r}}.$$

It can be shown that if $V^r(a_N) < \infty$ for any $r > 0$, then (a_N) is Cauchy and hence converges. Thus, by fixing $f_1,\dots,f_m : X \to \mathbf{C}$ and $x \in X$ and considering the sequence in N which is obtained, it suffices to prove that there exists $0 < r < \infty$ such that

$$V^r\left(\frac{1}{N}\sum_{n=1}^{N}\prod_{i=1}^{m} f_i(T^{\lfloor n^{c_i} \rfloor}(x))\right) < \infty$$

for almost every x. With r and $f_1,\dots,f_m$ fixed we obtain a function on X. To show that a function is finite almost everywhere it suffices to prove that its $L^p(X)$ norm is finite for some p. Our goal is therefore to prove a so-called variational inequality of the form

$$\left\| V^r\left(\frac{1}{N}\sum_{n=1}^{N}\prod_{i=1}^{m} f_i(T^{\lfloor n^{c_i} \rfloor}(x))\right)\right\|_{L^p(X)} \leq C\prod_{i=1}^{m}\|f_i\|_{L^{p_i}(X)}$$

for any $fi \in L^{p_i}(X)$ with $\frac{1}{p_1} + \dots + \frac{1}{p_m} = \frac{1}{p} \leq 1$.

The proof of this result will utilise methods both from pointwise ergodic theory (Krause, Mirek and Tao, 2022) and from additive combinatorics (Peluse, 2020). In particular, by a transference principle of Calderón (1968) it suffices to work in the case where $X = \mathbf{Z}$, $\mu(A)$ is the number of elements of A, and T is the shift map defined by $T(x) = x - 1$. In this case, the averages of study are

$$\frac{1}{N}\sum_{n=1}^{N}\prod_{i=1}^{m} f_i(x - \lfloor n^{c_i} \rfloor)$$

and one can use techniques from discrete harmonic analysis, such as the Fourier transform, to analyse these operators.

We will also obtain combinatorial applications along the way. Specifically, in addition to pointwise ergodic theory the additive combinatorial portion of the proof will also yield the following Peluse-type result. Fix non-integers $0 < c_1 < \dots < c_m$. There are constants $c,\ C > 0$, depending only on $c_1,\dots,c_m$, such that if $A \subseteq \{1,\dots,N\}$ lacks progressions of the form $x,\ x + \lfloor n^{c_1} \rfloor,\dots,x + \lfloor n^{c_m} \rfloor$ then

$$|A| \leq C\frac{N}{(\log\log N)^c}.$$

References

Bergelson, V. (1987) 'Weakly mixing PET', *Ergodic Theory and Dynamical Systems*, 7(3), pp. 337–349.

Birkhoff, G. (1931) 'Proof of the ergodic theorem', *Proceedings of the National Academy of Sciences*, 17(12), pp. 656–660.

Bourgain, J. (1990) 'Double recurrence and almost sure convergence', *Journal für die reine und angewandte Mathematik*, 404: pp. 140–161.

Bourgain, J. (1988a) 'On the maximal ergodic theorem for certain subsets of the integers', *Israel Journal of Mathematics*, 61(1), pp. 39–72.

Bourgain, J. (1988b) 'On the pointwise ergodic theorem on Lp for arithmetic sets', *Israel Journal of Mathematics*, 61(1), pp. 73–84.

Bourgain, J. (1989) 'Pointwise ergodic theorems for arithmetic sets', *Publications Math´ematiques de l'Institut des Hautes E´tudes Scientifiques*, 69(1), pp. 5–41.

Calderón, A. (1968) 'Ergodic theory and translation-invariant operators', *Proceedings of the National Academy of Sciences*, 59(2), pp. 349–353.

Donoso, S., Koutsogiannis A., Sun W. (2020) 'Pointwise multiple averages for sublinear functions'. *Ergodic Theory and Dynamical Systems*, 40(6), pp. 1594–1618.

Furstenberg, H. (1977) 'Ergodic behavior of diagonal measures and a theorem of szemer´edi on arithmetic progressions', *Journal d'Analyse Mathématique*, 31(1), pp. 204–256.

Furstenberg, H. (1990) 'Nonconventional ergodic averages', *Proceedings of the Symposium on Pure Mathematics*, 50: pp. 43–56.

Furstenberg, H. and Weiss B. (1996) 'A mean ergodic theorem for $ \frac{1}{N} \sum_{n=1} N f(Tn x)_s(T_{n^2} x)$', *Convergence in Ergodic Theory and Probability*, 5, pp. 193–228.

Host, B. and Kra B. (2005) 'Nonconventional ergodic averages and nilmanifolds', *Annuals of Mathematics:* pp. 397–488.

Krause, B. (2022) *Discrete Analogues in Harmonic Analysis: Bourgain, Stein, and Beyond*. American Mathematical Society, pp. 73–73.

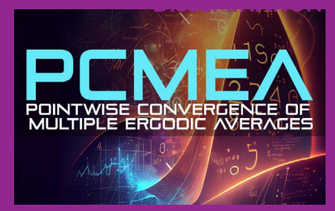
Krause, B., Mirek M. and Tao T. (2022) 'Pointwise ergodic theorems for non-conventional bilinear polynomial averages', *Annuals of Mathematics*, 195(3), pp. 997–1109.

Leibman, A (2005) 'Convergence of multiple ergodic averages along polynomials of several variables', *Israel Journal of Mathematics*, 146(1), pp. 303–315.

Peluse, S. (2020) 'Bounds for sets with no polynomial progressions', *Forum of Mathematics, Pi*, 8, e16.

von Neumann, J. (1932) 'Proof of the quasi-ergodic hypothesis', *Proceedings of the National Academy of Sciences*, 18(1), pp. 70–82.

Wierdl, M.(1989) *Almost everywhere convergence and recurrence along subsequences in ergodic theory*. PhD thesis, The Ohio State University, 1989.



PROJECT SUMMARY

Given an arbitrary measure preserving system we show that the multilinear ergodic averages sampled along an arbitrary number of sequences coming from a Hardy field converge pointwise almost everywhere. We aim to prove this for as wide a class of Hardy field functions as possible. To do so, we establish a long variational inequality along lacunary sequences which implies a maximal inequality, norm convergence, and pointwise convergence.

By a transference argument it suffices prove this long variational inequality in the case that the measure preserving system is the integers. This reduction allows us to use tools from discrete harmonic analysis, additive combinatorics, and analytic number theory. We then give applications in areas such as upcrossings, equidistribution, and combinatorics.

PROJECT LEAD

Max obtained his master's in mathematics (MMath) at the University of Warwick in 2021 and is now a third year PhD student studying under the supervision of Professor Ben Krause (Bristol University). His research interests lie in harmonic analysis (discrete harmonic analysis in particular), pointwise ergodic theory, and additive combinatorics.

PROJECT CONTACT

Max O'Keeffe

maximilian.okeeffe@kcl.ac.uk

https://www.kcl.ac.uk/people/maximilian-okeeffe

FUNDING

This research is funded by a studentship provided by the Heilbronn Institute for Mathematical Research.