Pointwise ergodic theorems for non-conventional bilinear polynomial averages along prime orbits

Background

Goldbach conjecture

In 1742, Goldbach wrote a letter to Euler proposing the following conjecture:

dass jede Zahl, welche aus zweyen numeris primis zusammengesetzt ist, ein aggregatum so vieler numerorum primorum sey, als man will (die unitatem mit dazu gerechnet), bis auf die congeriem omnium unitatum.

In modern mathematical language, he is asking whether

Every even integer can be written as sum of two odd primes.

This problem is still open despite much effort and partial progress. The first notable breakthrough was due to Vinogradov in the 1940s, in which he proved the so-called *ternary Goldbach conjecture*, namely that every *large enough* odd integer can be written as sum of three primes.

His method had two other side results that interested me.

- *Binary Goldbach:* He proved that *almost every* even integer, from the perspective of density, can be written as sum of two odd primes.
- *Ergodic theorem along primes:* He implicitly proved mean convergence of ergodic averages along the primes.

Although the binary Goldbach conjecture still seems to be far out of reach, even with current technology, there have been many important advances, two of which motivated my research in prime number theory.

1. Representing almost every even integer as a sum of two odd primes from a restricted (proper) subset of the primes; this line of research bears close relation with Green-Tao pseudorandomness theory, see for example (Shao, 2014b).
2. Obtaining Goldbach-type theorems for sums of, e.g. primes and semi-primes, or sums of several squares of primes; the most famous theorem in this direction is due to Chen (1973). This line of research has a close relationship to the Gauss circle problem and the Waring problem.

Although the statement of Goldbach's theorem is very accessible, progress on this line of inquiry has admitted far-reaching consequences in additive combinatorics and combinatorial number theory, encompassing the work of Green-Tao-Ziegler on detecting polynomial systems of equations inside the primes, most notably the *Green-Tao theorem*, Tao's proof of the *logarithmic Chowla conjecture*, and Yitang Zhang's *prime gaps theorem*.

These results, in particular, have shaped my research programme in prime number theory.

Pointwise erodic theorem

The study of pointwise convergence of ergodic averages dates back to Birkhoff (1931); it says that on any probability space (X, μ) equipped with a measure-preserving,[1] 'sufficiently randomising' transformation, $T : X \to X$,

(1) $$\frac{1}{N}\sum_{n\leq N} f(T^n x) \to \int_X f(\omega)\, d\mu(\omega)$$

almost surely, whenever $f \in L^\infty(X)$ is (say) bounded; informally: the 'time averages' of f converge to its 'spatial average.' This theorem was proven essentially concurrently with von Neumann's mean ergodic theorem (1932), which established norm convergence of (1).

A natural question, posed in the 1970s by Furstenberg and Bellow, is the existence of subsets $E \subset \mathbb{N}$ with 0 upper-Banach density,

$$\limsup_{|I|\to\infty \text{ an interval}} \frac{|E \cap I|}{|I|} = 0$$

so that

(2) $$\frac{1}{|[N] \cap E|}\sum_{n\leq N,\, n\in E} T^n f,$$

converged almost everywhere, initially for bounded functions f on a probability space; here and throughout, we use

$$[N] := \{1, \ldots, N\}$$

A natural candidate, first explored by Bourgain in the late eighties, concerned the case where $E = \mathbb{P}$ are the set of primes; he proved the following theorem.

Theorem 1 (Bourgain's prime ergodic theorem). *In the setting of Birkhoff's theorem, suppose additionally that all powers of T, $\{T^2, T^3, \ldots\}$ are sufficiently randomising. Then*

(3) $$\frac{1}{|[N] \cap \mathbb{P}|}\sum_{n\leq N,\, n\in \mathbb{P}} f(T^n x) \to \int_X f(\omega)\, d\mu(\omega)$$

almost surely, whenever $f \in L^\infty(X)$.[2]

Despite the simplicity of this statement, extensions of (3) have occupied the ergodic-theoretic community until the last decade; this line of inquiry was unable to be fully resolved by Bourgain—see Mirek and Trojan (2015), Mirek, Trojan and Zorin-Kranich (2017), and Wierdl (1988). In the meantime, Bourgain proved his famous polynomial ergodic theorems (Bourgain, 1988a, 1988b, 1989), which established pointwise convergence of the pertaining ergodic averages when $\mathbb{P}$ was replaced by a polynomial orbit, $\{P(1), P(2), \ldots\}$ for integer-valued P.

Motivated by Furstenberg's (1977) celebrated ergodic-theoretic proof of Szemerédi's theorem, Bourgain shortly there after proved that the bilinear averages

(4) $$\frac{1}{N}\sum_{n\leq N} f_1(T^{an}x) \cdot f_2(T^{bn}x), \quad a, b \in \mathbb{Z}$$

converge pointwise almost surely whenever $f_i \in L^\infty(X)$ are bounded (Bourgain, 1990)—and this remained essentially the state of the art progress on pointwise ergodic theorems for more than 30 years; crucial to this argument, and more generally to Furstenberg's, were the fact that the orbits in question were all *linear*.

The first breakthrough in the theory of multiple ergodic averages along non-linear polynomial orbits was due to H. Furstenberg and B. Weiss, (1996), in which norm convergence was established for the bilinear averages

(5) $$\frac{1}{N}\sum_{n\leq N} f_1(T^n x) \cdot f_2(T^{n^2} x);$$

this result admitted profound extensions, both dynamically and in combinatorial Ramsey theory, ultimately culminating in Miguel Walsh's (2012) ergodic theorem, an optimal result in the category of norm convergence of multiple ergodic averages along polynomial orbits.

The pointwise counterpart of Miguel Walsh's ergodic theorem remains in conjecture form, due to Furstenberg-Bergelson-Leibman, posed as Question 9 in V. Bergelson's (1996) survey on ergodic Ramsey theory:

Conjecture 1 (Furstenberg-Bergelson-Leibman conjecture, commutative case). *For any polynomials, $P_1, \ldots, P_m$, and commuting measure-preserving transformations, $T_1, \ldots, T_m : X \to X$, and bounded functions $f_1, \ldots, f_m \in L^\infty(X, \mu)$, the ergodic averages*

$$\frac{1}{N}\sum_{n\leq N}\prod_{i=1}^{m} T_i^{P_i(n)} f_i$$

converge μ-a.e.

This conjecture remains the 'holy grail' of pointwise ergodic theory; to give a sense of the difficulty, even understanding pointwise convergence of the simplest open bilinear case—involving a single measure-preserving transformation—namely, the pointwise convergence of (5), resisted significant efforts from Bourgain and others (Assani, 1998, 2005, 2010; Berend, 1985, 1988; Derrien and Lesigne, 1996; Donoso and Sun, 2018a, 2018b; Donoso, Koutsogiannis and Sun, 2020; Huang, Shao and Ye, 2019; Leibman, 2005; Lesigne, 1993; Lesigne, Rittaud and de la Rue, 2003). Indeed, pointwise convergence was not established until 2022, when Ben Krause

1. *That is, $\mu(T^{-1}E) = \mu(E)$ for every measurable E.*
2. *Convergence is guaranteed under the weaker assumption that $T : X \to X$ is merely measure-preserving, but the convergence to the spatial mean requires some further randomisation.*

(University of Bristol), Mariusz Mirek (Rutgers University) and Terence Tao (UCLA) (Krause, Mirek and Tao, 2022) proved the first joint extension of Bourgain's polynomial ergodic theorem (Bourgain, 1988a, 1988b, 1989) and bilinear ergodic theorem (Bourgain, 1990).

Current programme

Currently, my work on Goldbach and my study of pointwise ergodic theory have intertwined to produce my strongest result, jointly with Professor Krause, Professor Tao and Dr Joni Teräväinen (University of Cambridge). What follows is a brief summary of the relevant work in this direction, followed by an overview of KMTT (Krause-Mousavi-Tao-Teräväinen).

Goldbach conjecture

The departure point for my investigation of the density version of the Goldbach phenomenon was the work of Shao (2014a, 2014b); below, we use

$$d_*(P) := \liminf_{|I|\to\infty \text{ an interval}} \frac{|P\cap I|}{|I|}$$

to denote *lower* Banach density.

Theorem 2 *Let $P \subset \mathbb{P}$ with $d_*(P) > 5/8$. Then every sufficiently large odd integer can be written as $p_1 + p_2 + p_3$* with *$p_i \in P$. And, 5/8 is sharp.*

More recently he showed that there is no 'reasonable' density version of the binary Goldbach theorem.

Theorem 3 *For any $\epsilon > 0$ there exists a subset $A \subseteq \mathbb{P}$* with *$d_*(A) > 1 - \epsilon$, such that a positive proportion of the even positive integers, depending on ϵ, cannot be written as a sum of two primes in A.*

In-progress work with Michael Lacey (Georgia Tech), Yaghoub Rahimi (Georgia Tech), and Naga Manasa Venpati (LSU) concerns the behaviour of sum-sets of fairly dense subsets of primes.

Theorem 4 (Special Case). *Let $P \subset \mathbb{P}$ be a subset of primes with $d_*(P) > 0$. The P is an additive basis for $\mathbb{N}$, namely every* sufficiently large *integer, n, can be written as a sum of elements in $P + \cdots + P$ s-many times in the following regimes:*

- *3-fold sum-set: If $d_*(P) > 1/2$ and P is equidistributed* mod 100! *and n is odd then $n \in P + P + P$.*
- *4-fold sum-set: If $d_*(P) > 1/2$ and n is even then $n \in P + P + P + P$.*
- *s-fold sum-set: If $d_*(P) > \frac{s+2}{3s}$ and P* is *equidistributed* mod *$s!$, then $n \in P + \cdots + P$ is in the s-fold sum-set, provided that n and $s \geq 5$ share the same parity.*

By combining the techniques used to prove this theorem with discrete harmonic-analytic *compactness* arguments, I have established the following Goldbach-type result, which will appear in winter 2025; it can be viewed as a natural follow-up of the works of Matomäki (2008) and Teräväinen (2018).

Theorem 5 *Almost every large integer of the form $24k$+12 can be written as $a^2+b^2+p^2+1$ where $p \in \mathbb{P}$, $a^2 + b^2 + 1 \in \mathbb{P}$, and $(a, b) = 1$.*

Pointwise ergodic theory along the primes: combinatorial consequences

I currently own the sharpest results on pointwise convergence of ergodic averages along prime orbits (Giannitsi *et al.*, 2022; Lacey, Mousavi and Rahimi, 2022); the following two *quantitative visibility results* derive from this work and have proven instrumental as I prepare to consider more exotic, sparser, averages along the primes. The upshot of this work is as follows: 'small' subsets of intervals are generically 'invisible' along prime orbits.

Proposition 1 (Stucture theorem I). *Suppose that $F \subset I \subset \mathbb{Z}$ is a subset of an interval, I, with relative density $\delta \geq C_\epsilon \cdot |I|^{-\epsilon}$, $|F| = \delta|I|$. Then there exists a decomposition $I = I_1 \cup I_2$, where I_1, I_2 depend only on F, so that*

- *F is* barely visible *along prime orbits:*

$$\sup_{x\in I_1,\, N\geq 1} \frac{|\{p \leq N \text{ prime} : x - p \in F\}|}{N/\log N} \leq C \cdot \delta \cdot \log^2(1/\delta);$$

and

- *I_2 is small:* $|I_2| \leq \frac{|I|}{10}$.

This result is nearly optimal, in that it is known to fail without the logarithmic weighting (see: LaVictoire, 2011); conditional on the *generalised Riemann hypothesis*, we reduced the power of 2 to a single power of the logarithm, which is conjecturally sharp.

If we are willing to replace the logarithmic correction with a small power loss, we can in fact introduce a large degree of arithmetic uniformity, which has proved important in work on Theorem 9, and will likely impact joint extensions of Bourgain and KMTT, namely conjecture 2.

Proposition 2 (Structure theorem II, (Giannitsi *et al.*, 2022)). *Let $\epsilon > 0$ be arbitrarily small, and $R \geq 1$ be arbitrary. Then whenever I is an interval of length $|I| \geq 2^{C_\epsilon R^\epsilon}$, and $F \subset I$ is a subset of relative density $\delta \geq C_\epsilon |I|^{-\epsilon}$, one may decompose $I = I_1 \cup I_2$, so that*

- *F is* barely visible *along prime orbits in many arithmetic progressions: if we let Prob denote the uniform measure on arithmetic progressions of gap size $\leq R$ inside of I, then*

$$\text{Prob}\left(\sup_{x\in I_1,\, N\geq 1} \frac{|\{(x - \mathbb{P}_{b,y;N}) \cap F\}|}{|\mathbb{P}_{b,y;N}|} \leq C_\epsilon \cdot \delta^{1-\epsilon}\right) \geq 0.99$$

where

$$\mathbb{P}_{b,y;N} := \{p \leq N \textit{ prime} : p \equiv b \bmod y\};$$

- *I_2 is small: $|I_2| \leq C_\epsilon \cdot \delta^\epsilon \cdot |I|$.*

In other words, choosing from shifts of F and primes in any arithmetic progression are usually 'almost' independent events.

These two families of results left me well-situated for KMTT, which I will now describe.

Bilinear pointwise ergodic theory along the primes

In 2022, breakthrough work of Krause-Mirek-Tao established the following special case of the Furstenberg-Bergelson-Leibman conjecture.

Theorem 6 (Krause-Mirek-Tao, special case). *Suppose that $f_1, f_2 \in L^\infty(X, \mu)$ are bounded, and that $P \in \mathbb{Z}[\cdot]$ has degree ≥ 2. Then provided that $\{T, T^2, \ldots\}$ are 'sufficiently randomising,'*

$$\frac{1}{N}\sum_{n\leq N} f_1(T^n x) \cdot f_2(T^{P(n)} x) \to \int_X f_1(\omega)\, d\mu(\omega) \cdot \int_X f_2(\omega)\, d\mu(\omega)$$

converges almost surely to the product of expectations; μ-a.e. convergence is always guaranteed provided T is measure preserving.

Although this problem presents dynamically, the arguments lived at the interface of additive combinatorics and adelic harmonic analysis; the delicacy of these arguments has proven sufficiently restrictive that—prior to KMTT—no genuine extensions of Theorem 5 were known.

On the other hand, Teräväinen (2024) developed an additive combinatorial method to address the Furstenberg-Bergelson-Leibman averages weighted by the Möbius function.

Theorem 7 *Let (X, ν, T) be a measure-preserving system, μ be Möbius function, and $P_1, P_2, \cdots P_k$ be polynomials with integer coefficients. Let $f_1, f_2, \cdots f_k \in L^\infty(X)$. Then*

(6)

$$\lim_{N\to\infty} \frac{(\log N)^A}{N} \sum_{n<N} \mu(n) f(T^n x) \cdot f(T^{P_1(n)} x) \cdots f(T^{P_k(n)} x) = 0.$$

ν—almost everywhere.

While this result follows directly from the later work of Leng, the significance of Teräväinen's work is that it provided a general mechanism to address multiple ergodic averages weighted by 'pseudo-random' functions, e.g. the *von Mangoldt* function,

$$\Lambda(n) := \begin{cases} \log p & \text{if } n = p^\alpha,\ \alpha \geq 1 \\ 0 & \text{otherwise,} \end{cases}$$

which behaves like a (weighted) indicator of the set of primes

$$\frac{1}{N}\sum_{n\leq N} \Lambda(n) T^n f \cdot T^{P(n)} g \quad \text{``="} \quad \frac{1}{|[N] \cap \mathbb{P}|} \sum_{p\leq N,\, p\in\mathbb{P}} T^p f \cdot T^{P(p)} g.$$

These new techniques, combined with methods of KMT and appropriate additive combinatorial approximations, partially developed in the course of work on Goldbach, have allowed us to establish my strongest result thus far.

Theorem 8 (Special Case). *Suppose that $f_1, f_2 \in L^\infty(X, \mu)$ are bounded, and that $P \in \mathbb{Z}[\cdot]$ has degree ≥ 2. Then provided that $\{T, T^2, \ldots\}$ are 'sufficiently randomising,'*

$$\frac{1}{|[N] \cap \mathbb{P}|} \sum_{n \leq N,\, n \in \mathbb{P}} f_1(T^n x) \cdot f_2(T^{P(n)} x) \to \int_X f_1(\omega)\, d\mu(\omega) \cdot \int_X f_2(\omega)\, d\mu(\omega)$$

converges almost surely to the product of expectations; μ-a.e. convergence is always guaranteed, provided T is measure-preserving.

Compared to KMT, establishing **Theorem 8** required that we synthesise existing Ramsey-theoretic compactness arguments with the modern theory of *pseudo-random approximation*; in particular, *sieve theoretic techniques* were imported to the field for the first time. And, arithmetic considerations, unique to the set of primes, interposed in our adelic harmonic analysis, which required novel combinatorial arguments.

Future plans

In light of Bourgain's convergence result concerning (4), and the recent work of Matomäki and Tao on norm convergence of ergodic averages in 'short intervals', see (Matomäki *et al.*, 2023), KMTT admits two natural follow-ups, which I expect to pursue over the remainder of my fellowship.

Conjecture 2 *Suppose that $f_1, f_2 \in L^\infty(X, \mu)$ are bounded. Then, provided that T is* weakly mixing—very *randomising—*

$$\frac{1}{|[N] \cap \mathbb{P}|} \sum_{p \leq N,\, p \in \mathbb{P}} f_1(T^{ap} x) \cdot f_2(T^{bp} x) \to \int_X f_1(\omega)\, d\mu(\omega) \cdot \int_X f_2(\omega)\, d\mu(\omega), \quad a, b \in \mathbb{Z}$$

converges almost surely to the product of expectations; μ-a.e. convergence is always guaranteed, provided T is measure-preserving.

And

Conjecture 3 *Suppose that $f_1, f_2 \in L^\infty(X, \mu)$ are bounded, let $P \in \mathbb{Z}[\cdot]$ be a polynomial with integer coefficients, and let $\epsilon > 0$ be very small, possibly depending on P. Then, provided that T is weakly mixing*

$$\frac{1}{N^{1-\epsilon}} \sum_{N \leq n < N + N^{1-\epsilon}} f_1(T^n x) \cdot f_2(T^{P(n)} x) \to \int_X f_1(\omega)\, d\mu(\omega) \cdot \int_X f_2(\omega)\, d\mu(\omega), \quad N \in 2^{2^N}$$

converges almost surely to the product of expectations along sparse sequences; μ-a.e. convergence is always guaranteed, provided T is measure-preserving.

In the previous conjecture, the restriction to the sparse set of times is in fact essentially necessary.

References click here

PROJECT SUMMARY

Following our proof of the first Goldbach result in number field setting after 60 years (Mitsui 1960s), We give new information on density ternary Goldbach problem after 10 years (Shao 2013), and also design a new strategy to prove density version of representing a number as sum of "certain" primes. We also apply Rosser Sieve arguments to prove represtation of integers into sum of primes coming from a zero density subset.

From Ergodic theory side, we give the first pointwise bilinear theorem along the primes, which is the natural pointwise convergence along prime result after Wierdl (1990) and a natural follow up after recent pointwise bilinear break-through by Krause-Mirek-Tao (2021). In another project, the sharpest pointwise convergence average along primes for "near" integrable function is given, which is sharp up to Generalized Riemann Hypothesis.

PROJECT LEAD

Dr Seyyed Hamed Mousavi received his PhD from Georgia Tech in 2022, under the advisement of Professor E. Croot. Dr Mousavi's current academic position, held from autumn 2022, is an EPSRC Postdoctoral Fellowship, funded by Professor Krause's New Investigator Grant on pointwise ergodic theory.

Dr Mousavi's interests span prime number theory and pointwise ergodic theory. His initial training was in analytic number theory, which he has complemented over the last four years by developing a facility with techniques and ideas from both harmonic analysis and (additive) combinatorics.

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