Arithmetic properties for sparse sequence of integers and its application in number theory, additive combinatorics and ergodic theory

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The famous prime number theorem tells us that the number of primes up to x is asymptotically equal to $x/\log x$. In 1953, Piatetski-Shapiro (1953) first considered the primes of the form $\lfloor n^c \rfloor$ for some $1 < c < 2$, which today are known as the *Piatetski-Shapiro primes*.

Let $\mathbb{P}$ be the set of primes and

$$\mathbb{N}_c := \{\lfloor n^c \rfloor : n \in \mathbb{N}\}, \text{ with } 1 < c < 2;$$

let

(1) $$\mathbb{P}_c := \mathbb{P} \cap \mathbb{N}_c.$$

Piatetski-Shapiro proved that for $1 < c < 1.1$,

(2) $$\pi_c(x) := \sum_{\substack{p \in \mathbb{P}_c \\ p \le x^c}} 1 = (1 + o(1)) \frac{x}{c \log x}.$$

The range of c has been improved significantly, and the best result to date is due to Rivat and Sargos (2001), who showed that (2) holds for $1 < c < 2817/2426 \approx 1.161$. On the other hand, if one is only interested in bounding the growth rate of $\pi_c(x)$, the best range for c is due to Rivat and Wu (2001), who proved that

$$\pi_c(x) \ge \text{Const} \cdot \frac{x}{\log x},$$

for $$1 < c < \frac{243}{205} \approx 1.185.$$

Much of my research to date has been focused on investigating the extent to which the following two fundamental theorems of additive number theory can be extended to (1):

1. **(Vinogradov's theorem)** Every sufficiently large odd integer can be written as a sum of three primes. This result was proved by Vinogradov.
2. **(Roth's theorem for primes)** Any subset of primes with positive upper density contains a non-trivial 3-term arithmetic progression. This was established for primes by Green (2005).

In 1992, Balog and Friedlander (1992) first established Vinogradov's theorem for Piatetski-Shapiro primes with $1 < c < 1.05$. In 2014, Mirek (2015a) proved a **Roth-type theorem** for such primes when $1 < c < 72/71 \approx 1.014$.

The best ranges for c in extensions of both theorems are due to myself, jointly with Shanshan Du and Hao Pan (2019):

Theorem 3 (Vinogradov extension). *For any* $c_1, c_2, c_3 \in (1, 41/35)$, *every sufficiently large odd integer* N *can be represented as*

$$N = p_1 + p_2 + p_3,$$

where $p_i \in P_{c_i}$, $1 \le i \le 3$.

Theorem 4 (Roth-type extension). *For any* $c \in (1, 243/205)$, *every subset of* $E \subset \mathbb{P}_c$ *with positive upper density*

$$\limsup_{|I| \to \infty \text{ an interval}} \frac{|E \cap I|}{|I|} > 0$$

contains a non-trivial 3-term arithmetic progression.

In particular, we established:

- Vinogradov's extension: $1 < c < 41/35 \approx 1.171$ for each exponent
- Roth-type extension: $1 < c < 243/205 \approx 1.185$.

This line of inquiry admits natural extensions to pointwise ergodic theory, which I have recently begun exploring. The departure point for my work was Rosenblatt and Wierdl's (1995) conjecture, formulated in 1995; below, a *measure-preserving system*,

$$(X, \Sigma, \mu, T),$$

is a probability space equipped with a measure-preserving transformation,

$$T : X \to X, \ \mu(T^{-1}E) = \mu(E) \text{ for all } E \in \Sigma.$$

Conjecture 1. *Suppose* $\{a_n\}_{n=1}^{\infty}$ *has zero Banach density,*

$$\limsup_{|I| \to \infty \text{ an interval}} \frac{|\{a_n\} \cap I|}{|I|} = 0,$$

and that (X, Σ, μ, T) *is aperiodic measure-preserving system:*

$$\mu(\{x : T^n x = x \text{ for some } n \in \mathbb{N}\}) = 0.$$

Then there exists $f \in L^1(X)$ *such that*

$$\frac{1}{N} \sum_{n=1}^{N} T^{a_n} f$$

does not converge almost everywhere.

This conjecture was disproved by Buczolich (2007). Subsequently, Urban and Zienkiewicz (2007) proved that for $1 < c < 1.001$, any measure-preserving system (X, Σ, μ, T), and any $f \in L^1(X)$, the ergodic averages

$$M_{N,c} f = \frac{1}{N} \sum_{n \le N} T^{\lfloor n^c \rfloor} f$$

converge almost everywhere. The range of c was later improved to $1 < c < 30/29 \approx 1.034$ by Mirek (2015b), which remained the state of the art for 12 years.

Recently, in collaboration with Professor Krause (2025), we established the following theorem as a special case of a more general phenomenon.

Theorem 5. *For any* $1 < c < 8/7 \approx 1.1429$, *any measure-preserving system* (X, Σ, μ, T), *and any* $f \in L^1(X)$, *the averages* $M_{N,c}f$ *converge almost everywhere.*

In the same paper, we also proved:

Theorem 6. *For any* $1 < c < 2$, *any measure-preserving system* (X, Σ, μ, T), *and any* $f \in L^p(X)$ *with* $p > 1$, *the averages* $M_{N,c}f$ *converge almost everywhere.*

We also expect to prove the following convergence result: multi-linear extensions are also present.

Theorem 7. *For any measure-preserving system* (X, Σ, μ, T) *and* $f \in L^p(X)$ *with* $p > 1$, *the averages*

$$\frac{1}{\pi_c(N)} \sum_{p \in \mathbb{P}_c, p \leq N^c} T^p f$$

converge almost everywhere, see (2).

Back in the additive combinatorial setting, a natural future direction is to investigate whether $\mathbb{P}_c$ contain arbitrarily long arithmetic progressions. In 2019, Li and Pan established:

Theorem 8 (Li and Pan). *For any* $m \geq 3$ *and* $1 < c < \frac{1}{1-2^{-2^{m^2 4^m}}}$, *the set* $\mathbb{P}_c$ *contains non-trivial* m*-term arithmetic progressions.*

For $m = 3$, their range $1 < c < \frac{1}{1-2^{-2359296}}$ is significantly smaller than the range $1 < c < 243/205 \approx 1.185$ in Theorem 4.

This suggests that substantial improvement may be possible.

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POINTWISE CONVERGENCE OF MULTIPLE ERGODIC AVERAGES

PROJECT SUMMARY

In this project we investigate dynamical and arithmetic statistics of sequences of the form $\{\lfloor nc \rfloor\}$, for 1<c<2 as large as possible, as well as the statistics of the intersection of these sequences with the primes.

PROJECT LEAD

Yu-Chen Sun is an ERC-sponsored postdoctoral fellow, working under the supervision of Professor Ben Krause (Bristol University). He completed his PhD in 2024 under the advisement of Kaisa Matomaki (University of Turku).

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